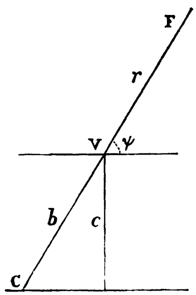
SOLUTIONS OF EXERCISES.

40

THE VERTEX of an hyperbola and one asymptote is fixed. Find the locus of the focus.

SOLUTION.



Let r, ψ be the polar co-ordinates of the focus, referred to the vertex and to a line through the vertex parallel to the asymptote,

c, b, the distances of the vertex from the asymptote and from the centre,

$e = \sec\psi$, the eccentricity.

Then, as may easily be seen,

$$b \sin\psi = c,$$
$$r = b(\sec\psi - 1);$$
$$\therefore\ r\sin\psi = c(\sec\psi - 1),$$

or

$$r = c\tan\tfrac{1}{2}\psi \,.\, \sec\psi,$$

the equation to the locus sought. Expressed in rectangular co-ordinates, this becomes, after reduction,

$$x^2(y + 2c) = c^2 y,$$

a cubic, symmetrical with the axis of y. The curve has three branches A, B, C and three asymptotes a, b, c, of which a is parallel to the axis of x, and b and c are parallel to the axis of y; a is common to B and C, b to A and C, c to A and B. [*Ormond Stone.*]

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ABC is an equilateral triangle inscribed in a circle. From any point P within the triangle BAC, straight lines PA, PB, PC are drawn. Show, *geometrically*, that $PB + PC - PA$ is a minimum when P is on the circumference of the circle about ABC. [*R. D. Bohannan.*]

SOLUTION.

Let P_1, P_2 be two points near the point P, and so selected that $CP_1 = CP_2$. From the nature of minimum values we have

$$P_1B + P_1C - P_1A = P_2B + P_2C - P_2A;$$
$$\therefore\ P_1B - P_1A = P_2B - P_2A;$$

$\therefore$ P_1, P_2 are on an hyperbola having A, B for foci. Letting P_1, P_2 come indefinitely near P, P_1P_2 is the tangent to the hyperbola; but, since $CP_1 = CP_2$

P_1P_2 is perpendicular to CP, PC is normal to the hyperbola, and bisects the angle between PB and PA prolonged.

In like manner, selecting P_1, P_2 so that $P_1A = P_2A$, we have

$$P_1B + P_1C = P_2B + P_2C;$$

$\therefore$ P is on an ellipse whose foci are B, C. P_1P_2 is tangent to this ellipse; $\therefore$ P_1A is normal; $\therefore$ PA bisects the angle BPC.

Thus, calling PD the prolongation of CP, and PE the prolongation of AP, we have

$$\angle APB = \angle APC = \angle DPE = \angle DPB;$$
$$\therefore \angle APB = 60^\circ;$$

$\therefore$ P moves on a circle about ABC. [*R. D. Bohannan.*]

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CIRCLES of given radius are drawn through the focus of a fixed parabola, cutting the curve in four points. Show that the products of the focal radii to these points are all equal. [*W. M. Thornton.*]

SOLUTION I.

The equation of the parabola is

$$\frac{l}{\rho} = 1 - \cos\vartheta.$$

Let a, α be the co-ordinates of the centre of the circle; its equation will be

$$\rho = 2a\cos(\vartheta - \alpha).$$

For common points

$$\rho = \frac{2a}{\rho}\left[(\rho - l)\cos\alpha + \sin\alpha\sqrt{(2\rho l - l^2)}\right].$$

Clear of fractions, transpose, and square, and we have a quartic in ρ, of which the absolute term is $4a^2l^2$;

$$\therefore \rho_1\rho_2\rho_3\rho_4 = 4a^2l^2,$$

and is independent of α. [*Chas. Puryear.*]

[Prof. Graves directs attention to the fact that this property is given in Smith's *Conic Sections*. *Pereant qui nostra ante nos dixerunt!* By an analagous proof it is easy to show that the relation holds for all conics. Prof. Bohannan gives the same proof as above, and shows that the sum of the reciprocals of the focal radii is also constant.—*W. M. T.*]

SOLUTION II.

The equation of a parabola, referred to the axis and directrix, is

$$y^2 = 4p(x - p). \tag{1}$$

The equation of a circle, radius r, through the focus $(2p, 0)$ is

$$(x-a)^2+(y-b)^2=(2p-a)^2+b^2=r^2. \qquad (2)$$

From (1) and (2) we find

$$x^2+2(2p-a)x-4p(2p-a)=2by. \qquad (3)$$

Squaring and eliminating y^2,

$$x^4+4(2p-a)x^3-4a(2p-a)x^2-16pr^2+16p^2r^2=0. \qquad (4)$$

If x_1, x_2, x_3, x_4 are the roots of equation (4), the product of focal radii to the intersections of (1) and (2) is

$$x_1x_2x_3x_4=16p^2r^2, \qquad (5)$$

which is constant, if p and r are constant.

Corollary. We also have

$$x_1x_2x_3+x_2x_3x_4+x_3x_4x_1+x_4x_1x_2=16pr^2. \qquad (6)$$

Dividing by (5)

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=\frac{1}{p}. \qquad (7)$$

Therefore, if circles be drawn through the focus of a fixed parabola cutting the curve in four points, the sum of the reciprocals of the focal radii to these points is constant. [*Wm. E. Heal.*]

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Find the condition that the cubic

$$6x^3-(2n+8)x^2+n(n+1)x+n(n+1)(2-n)=0$$

may have equal roots

SOLUTION I.

The discriminant of $a_0x^3+3a_1x^2+3a_2x+a_3=0$ is known to be*

$$\begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix};$$

$$\begin{vmatrix} 18 & -4(n+4) & n(n+1) & 0 \\ 0 & 18 & -4(n+4) & n(n+1) \\ -2(n+4) & 2n(n+1) & 3n(n+1)(2-n) & 0 \\ 0 & -2(n+4) & 2n(n+1) & 3n(n+1)(2-n) \end{vmatrix}=0$$

is the required condition.

*See Burnside and Panton's *Theory of Equations*, p. 305. Eliminant of $a_0x^2+2a_1x+a_2=0$ and $a_1x^2+2a_2x+a_3=0$ (Dialytic Method).

Drop successively the factors n, $(n+1)$, $(n-2)$, $(n-1)$ along with other reductions, as follows:—

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -(n+4) & n(n+1) & 3n(n+1)(2-n) & 0 \\ 0 & -(n+4) & 2n(n+1) & 3(2-n) \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -(n+4) & n(n+1) & 3n(n+1)(2-n) & 0 \\ -18 & 3(n+4) & 0 & 3(2-n) \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -n+2 & n^2-4 & 3n(n+1)(2-n) & n-2 \\ -6 & n+4 & 0 & 2-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 9 & -2(n+4) & n(n+1) & 0 \\ 0 & 9 & -4(n+4) & 1 \\ -1 & n+2 & -3n(n+1) & 1 \\ -6 & n+4 & 0 & 2-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 0 & 7n+10 & -26n(n+1) & 9 \\ 0 & 9 & -4(n+4) & 1 \\ -1 & n+2 & -3n(n+1) & 1 \\ 0 & -5n-8 & 18n(n+1) & -4-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n+10 & -13n(n+1) & 9 \\ 9 & -2(n+4) & 1 \\ -5n-8 & 9n(n+1) & -4-n \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+5n+72 & 0 \\ 9 & -2(n+4) & 1 \\ 4n+28 & 7n^2-7n-32 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+5n+72 \\ 4n+28 & 7n^2-7n-32 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n^2+12n+1 \\ 4n+28 & 7n^2-3n-4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 7n-71 & -13n-1 \\ 4n+28 & 7n+4 \end{vmatrix} = 0,$$

or $$101n^2 - 101n - 256 = 0;$$
therefore the required condition is $n = -1$, or $= 0$, or $= 1$, or $= 2$; or $101n^2 - 101n - 256 = 0$.

The last condition, however, would give to the cubic irrational co-efficients.

[*R. H. Graves.*]

SOLUTION II.

The cubic $$Ax^3 + Bx^2 + Cx + D = 0$$
will have equal roots, if its discriminant
$$27A^2D^2 + 4AC^3 + 4DB^3 - B^2C^2 - 18ABCD$$
$$= D(27A^2D + 4B^3 - 18ABC) + 4AC^3 - B^2C^2$$
vanish. In the given equation $4AC^3 - B^2C^2$ is easily seen to be divisible by D, and therefore one condition of equal roots is
$$D = 0,$$
which is satisfied by $n = 0$, $n = -1$, $n = 2$.

Dividing by D and equating the remaining factor to zero, we have
$$101n^3 - 202n^2 - 155n + 256 = 0.$$
One root is evidently $n = 1$, and we easily find the others to be
$$n = \tfrac{1}{2} + \tfrac{15}{202}\sqrt{505}, \quad n = \tfrac{1}{2} - \tfrac{15}{202}\sqrt{505}.$$

[*Wm. E. Heal.*]

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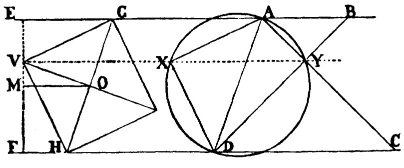
Construct a square; given one vertex and two parallel lines on which the extremities of the opposite diagonal are located.

SOLUTION I.



Let V be the vertex, and AB, CD the parallel lines given. Let fall from V on AB, CD the perpendiculars VE, VF. On the same side of EF, when V is between the parallels, otherwise on opposite sides, on AB lay off $EG = VF$, and on CD lay off $FH = VE$. G and H will be the extremities of the opposite diagonal, whence the square may be easily constructed.

[*Ormond Stone.*]

[Solved in the same way substantially by R. D. Bohannan and Chas. Puryear.]

SOLUTION II.

Through the given point V draw a straight line VX, parallel to the given lines AB, CD. At any point Y, of VX, draw two lines YAC, YBD, each making an angle of 45° with VX. Through AYD pass a circle cutting VX in X. Then

$AX = DX$ and they are at right angles to each other. Lines drawn through V parallel to AX, DX will be sides of the required square.

[*R. D. Bohannan.*]

SOLUTION III.

Draw VM perpendicular to the parallel midway between the given parallels, and on the latter lay off MO equal to half the distance between the given parallels. GOH perpendicular to OV is the required diagonal.

[*W. M. Thornton.*]

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GENERALIZATION.

A, B, C, etc. are the points of application of a system of parallel forces x, y, z, etc.; Q is their centre; ρ is their resultant; p, q, r, etc. are the radius vectors of A, B, C, etc., measured from an arbitrary origin M; $u = AB$, etc.

Show that if $MQ = h$

$$h^2 = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma u^2 xy}{\rho^2}.$$

SOLUTION.

Let $d_1 = QA$, $d_2 = QB$, etc.

It may be shown, as in articles 138 and 139 of Todhunter's *Analytical Statics*, that
$$\Sigma p^2 x = h^2 \rho + \Sigma d_1^2 x \quad \text{and} \quad \rho \Sigma d_1^2 x = \Sigma u^2 xy;$$

$$\therefore\ h^2 = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma d_1^2 x}{\rho} = \frac{\Sigma p^2 x}{\rho} - \frac{\Sigma u^2 xy}{\rho^2}.$$

[*R. H. Graves.*]

82

THE major axes of two similar and equal concentric ellipses intersect at right angles, and the area common to the two curves is half that of either ellipse. Find the eccentricity.

[*Ormond Stone.*]

SOLUTION.

The conditions of the problem give the equation,

$$\tfrac{1}{2}a^2b^2 \int_0^{\frac{1}{4}\pi} \frac{d\vartheta}{a^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta} = \tfrac{3}{16}\pi ab,$$

in which the usual notation is employed. Whence, by reduction,

$$b / a = \tan \tfrac{1}{8}\pi = \sqrt{2} - 1;$$

$$\therefore\ e = \sqrt{\left(\frac{a^2 - b^2}{a^2}\right)} = \sqrt{[2(\sqrt{2} - 1)]}.$$

[*R. H. Graves.*]